Some problems in the axiomatic foundation of geometry

Victor Pambuccian

Arizona State University, USA

April 3, 2021

The parallel postulate

- There is at most one line parallel to a given line passing through a given point.

The parallel postulate

- There is at most one line parallel to a given line passing through a given point.
- Every triangle has a center of the circumscribed circle.

The parallel postulate

- There is at most one line parallel to a given line passing through a given point.
- Every triangle has a center of the circumscribed circle.
- Through any point inside a given angle there exists a line which intersects both legs of the angle.

Absolute geometry $\mathcal{A}$

- The language $L_{B\equiv}$ in which the axiom system is expressed has only one sort of individual variables, to be referred to as *points*, as well as two relation symbols, a ternary one B, for *betweenness*, with $B(abc)$ to be read as ‘point b lies between a and c’ (and b may be a or c, and $a = b = c$ is also allowed), a quaternary one $\equiv$ for *equidistance*, with $ab \equiv cd$ to be read ‘b is as distant from a as d is from c’ (or ‘ab is congruent to cd’) The axioms are:

Absolute geometry

- $ab \equiv pq \wedge ab \equiv rs \rightarrow pq \equiv rs$,

Absolute geometry

- $ab \equiv pq \wedge ab \equiv rs \to pq \equiv rs$,
- $ab \equiv cc \to a = b$

Absolute geometry

- $ab \equiv pq \wedge ab \equiv rs \to pq \equiv rs$,
- $ab \equiv cc \to a = b$
- [Segment transport axiom]$(\forall qabc)(\exists x)\, B(qax) \wedge ax \equiv bc$

Absolute geometry

- $ab \equiv pq \wedge ab \equiv rs \rightarrow pq \equiv rs$,
- $ab \equiv cc \rightarrow a = b$
- [Segment transport axiom]$(\forall qabc)(\exists x)\, B(qax) \wedge ax \equiv bc$
- [Five segment axiom (a variant of "side-angle-side")]
  $B(abc) \wedge B(a'b'c') \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ad \equiv a'd'$
  $\wedge bd \equiv b'd' \rightarrow (dc \equiv c'd' \vee a = b)$

Absolute geometry

- $ab \equiv pq \wedge ab \equiv rs \rightarrow pq \equiv rs$,
- $ab \equiv cc \rightarrow a = b$
- [Segment transport axiom]$(\forall qabc)(\exists x)\, B(qax) \wedge ax \equiv bc$
- [Five segment axiom (a variant of "side-angle-side")]
  $B(abc) \wedge B(a'b'c') \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ad \equiv a'd'$
  $\wedge bd \equiv b'd' \rightarrow (dc \equiv c'd' \vee a = b)$
- $B(aba) \rightarrow a = b$

Absolute geometry

- $ab \equiv pq \wedge ab \equiv rs \rightarrow pq \equiv rs$,
- $ab \equiv cc \rightarrow a = b$
- [Segment transport axiom]$(\forall qabc)(\exists x)\, B(qax) \wedge ax \equiv bc$
- [Five segment axiom (a variant of "side-angle-side")] $B(abc) \wedge B(a'b'c') \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ad \equiv a'd'$ $\wedge bd \equiv b'd' \rightarrow (dc \equiv c'd' \vee a = b)$
- $B(aba) \rightarrow a = b$
- [The Pasch axiom]$(\forall abcde)(\exists f)\, B(cea) \wedge B(dba)$ $\rightarrow B(efd) \wedge B(bfc)$,

Absolute geometry

- $ab \equiv pq \wedge ab \equiv rs \to pq \equiv rs$,
- $ab \equiv cc \to a = b$
- [Segment transport axiom]$(\forall qabc)(\exists x)\, B(qax) \wedge ax \equiv bc$
- [Five segment axiom (a variant of "side-angle-side")] $B(abc) \wedge B(a'b'c') \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ad \equiv a'd'$ $\wedge bd \equiv b'd' \to (dc \equiv c'd' \vee a = b)$
- $B(aba) \to a = b$
- [The Pasch axiom]$(\forall abcde)(\exists f)\, B(cea) \wedge B(dba)$ $\to B(efd) \wedge B(bfc)$,
- [Lower-dimension axiom]$(\exists abc)\, \neg(B(abc) \vee B(bca) \vee B(cab))$,

Absolute geometry

- $ab \equiv pq \wedge ab \equiv rs \rightarrow pq \equiv rs$,
- $ab \equiv cc \rightarrow a = b$
- [Segment transport axiom]$(\forall qabc)(\exists x)\, B(qax) \wedge ax \equiv bc$
- [Five segment axiom (a variant of "side-angle-side")] $B(abc) \wedge B(a'b'c') \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ad \equiv a'd' \wedge bd \equiv b'd' \rightarrow (dc \equiv c'd' \vee a = b)$
- $B(aba) \rightarrow a = b$
- [The Pasch axiom]$(\forall abcde)(\exists f)\, B(cea) \wedge B(dba) \rightarrow B(efd) \wedge B(bfc)$,
- [Lower-dimension axiom]$(\exists abc)\, \neg(B(abc) \vee B(bca) \vee B(cab))$,
- [Upper-dimension axiom]$u \neq v \wedge au \equiv av \wedge bu \equiv bv \wedge cu \equiv cv \rightarrow (B(abc) \vee B(bca) \vee B(cab))$.

Algebraic characterization of models of absolute geometry

- Based on the work of Friedrich Bachmann, Wolfgang Pejas provided in 1961 an algebraic characterization of models of absolute geometry. The models are also called Hilbert planes.

Two main weakenings of the parallel postulate

- Axiom **R**: There exists a rectangle. Equivalently: The sum of the angles of every triangle (or just of one triangle) is 180°.

Two main weakenings of the parallel postulate

- Axiom **R**: There exists a rectangle. Equivalently: The sum of the angles of every triangle (or just of one triangle) is 180°.
- Max Dehn (1900) showed that **R** is strictly weaker than **P**, the Parallel Postulate.

Two main weakenings of the parallel postulate

- Axiom **R**: There exists a rectangle. Equivalently: The sum of the angles of every triangle (or just of one triangle) is 180°.
- Max Dehn (1900) showed that **R** is strictly weaker than **P**, the Parallel Postulate.
- There is no statemet σ in terms of the betweenness predicate B alone, such that $\mathcal{A} \vdash \mathbf{R} \leftrightarrow \sigma$.

Two main weakenings of the parallel postulate

- Axiom **R**: There exists a rectangle. Equivalently: The sum of the angles of every triangle (or just of one triangle) is 180°.
- Max Dehn (1900) showed that **R** is strictly weaker than **P**, the Parallel Postulate.
- There is no statemet σ in terms of the betweenness predicate B alone, such that $\mathcal{A} \vdash \mathbf{R} \leftrightarrow \sigma$.
- A further weakening of **P**, an axiom weaker than **R**, is the *Lotschnittaxiom* **L** studied by Bachmann (1964).

Two main weakenings of the parallel postulate

- Axiom **R**: There exists a rectangle. Equivalently: The sum of the angles of every triangle (or just of one triangle) is 180°.
- Max Dehn (1900) showed that **R** is strictly weaker than **P**, the Parallel Postulate.
- There is no statemet σ in terms of the betweenness predicate B alone, such that $\mathcal{A} \vdash \mathbf{R} \leftrightarrow \sigma$.
- A further weakening of **P**, an axiom weaker than **R**, is the *Lotschnittaxiom* **L** studied by Bachmann (1964).
- **L** states that: The perpendiculars on the sides of a right angle intersect each other.

Two main weakenings of the parallel postulate

- Axiom **R**: There exists a rectangle. Equivalently: The sum of the angles of every triangle (or just of one triangle) is 180°.
- Max Dehn (1900) showed that **R** is strictly weaker than **P**, the Parallel Postulate.
- There is no statemet σ in terms of the betweenness predicate B alone, such that $\mathcal{A} \vdash \mathbf{R} \leftrightarrow \sigma$.
- A further weakening of **P**, an axiom weaker than **R**, is the *Lotschnittaxiom* **L** studied by Bachmann (1964).
- **L** states that: The perpendiculars on the sides of a right angle intersect each other.
- Equivalent formulation: Through any point inside a right angle there exists a line which intersects both legs of the angle.

Additional equivalents of the Lotschnittaxiom

- Lagrange (1802): If the lines a and b are two intersecting lines that are parallel to a line g, then the reflection of a in b is also parallel to g.

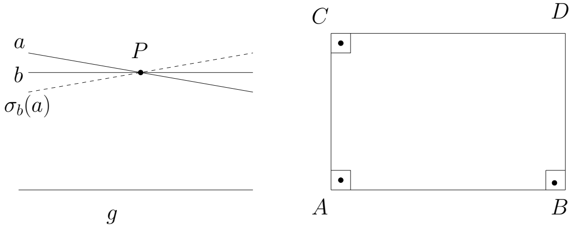

Figure: Lagrange's axiom and the *Lotschnittaxiom*

Proved equivalent to the Lotschnittaxiom with Celia Schacht (2019)

- A. Lippman (1906): Given any circle, there exists a triangle containing that circle in its interior.

Proved equivalent to the Lotschnittaxiom with Celia Schacht (2019)

- A. Lippman (1906): Given any circle, there exists a triangle containing that circle in its interior.
- Henri Lebesgue (1936): Given any convex quadrilateral, there exists a triangle containing that convex quadrilateral in its interior.

A universal statement equivalent to the Lotschnittaxiom

- VP (1994): In an isosceles triangle with base angle of 45°, the altitude is less than or equal than the base

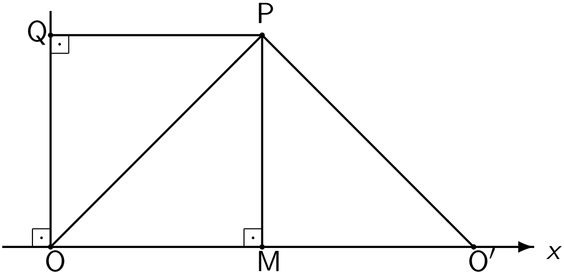

Figure: $MO \equiv MO'$, $PM \equiv PQ$ and $PM \leq OO'$

Aristotle's Axiom

- Aristotle's axiom appears in Aristotle's *On the Heavens*.

Aristotle's Axiom

- Aristotle's axiom appears in Aristotle's *On the Heavens*.
- The following arguments make it plain that every body which revolves in a circle must be finite. If the revolving body be infinite, the straight lines radiating from the centre will be infinite. But if they are infinite, the intervening space must be infinite. "Intervening space" I am defining as space beyond which there can be no magnitude in contact with the lines. This must be infinite. In the case of finite lines it is always finite, and moreover it is always possible to take more than any given quantity of it, so that this space is infinite in the sense in which we say that number is infinite, because there exists no greatest number.

Aristotle's Axiom

- A more precise statement can be found in Proclus:
- To anyone who wants to see this argument constructed, let us say that he must accept in advance such an axiom as Aristotle used in establishing the finiteness of the cosmos: If from a single point two straight lines making an angle are produced indefinitely, the interval between them when produced indefinitely will exceed any finite magnitude.

Aristotle's Axiom

- **Ar** If $\widehat{XOY}$ is an acute angle and AB is any segment, then there exists a point P on the ray $\overrightarrow{OY}$ and a point Q on the ray $\overrightarrow{OX}$, such that PQ is perpendicular to OX and $PQ > AB$.

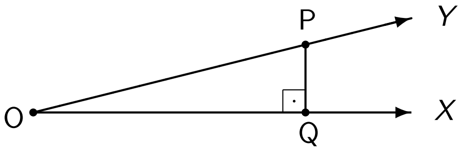

Figure: Aristotle's axiom

Aristotle's Axiom

- **Ar** is easily seen to be a consequence of **P**, as one simply draws a parallel to Ox at distance $2AB$, and its intersection with Oy delivers the desired point.

Aristotle's Axiom

- **Ar** is easily seen to be a consequence of **P**, as one simply draws a parallel to Ox at distance $2AB$, and its intersection with Oy delivers the desired point.
- **Ar** can also be seen as an elementary version of the Archimedean axiom. If one thinks that, given an acute angle $\widehat{XOY}$ and points M and N on the rays $\overrightarrow{OY}$ respectively $\overrightarrow{OX}$ such that MN is perpendicular to $\overrightarrow{OYX}$, then one can find points P and Q on $\overrightarrow{OY}$ respectively $\overrightarrow{OX}$, with PQ perpendicular $\overrightarrow{OYX}$, such that $PQ > 2MN$, then one can easily see how Archimedeanity implies **Ar**. It is true, in absolute geometry, that there exists P and Q with that property, but I do not know of any synthetic proof of that fact (the only proof I know (VP, 2019) uses Pejas's algebraic characterization fo Hilbert planes).

Aristotle's Axiom

- **Ar** If $\widehat{XOY}$ is an acute angle and AB is any segment, then there exists a point P on the ray $\overrightarrow{OY}$ and a point Q on the ray $\overrightarrow{OX}$, such that PQ is perpendicular to OX and $PQ > AB$.

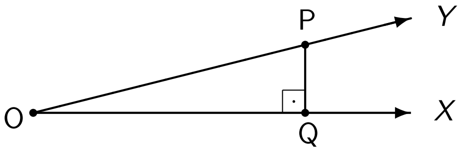

Figure: Aristotle's axiom

$\mathbf{P} \Leftrightarrow \mathbf{Ar} + \mathbf{L}$

- (VP, 1994) $\mathcal{A} \vdash \mathbf{P} \Leftrightarrow \mathbf{Ar} + \mathbf{L}$

$\mathbf{P} \Leftrightarrow \mathbf{Ar} + \mathbf{L}$

- (VP, 1994) $\mathcal{A} \vdash \mathbf{P} \Leftrightarrow \mathbf{Ar} + \mathbf{L}$
- This represents a splitting of **P** into two strictly weaker axioms. Their disjunction ($\mathbf{Ar} \vee \mathbf{L}$) is not provable in absolute geometry $\mathcal{A}$.

Pure incidence statements equivalent to **L**

- **ML**: Given three parallel lines, there is a line that intersects all three of them.

Pure incidence statements equivalent to **L**

- **ML**: Given three parallel lines, there is a line that intersects all three of them.
- $(\forall abc)(\exists u)\ \times(ab) \vee \times(bc) \vee \times(ca) \vee (\times(au) \wedge \times(bu) \wedge \times(cu))$

Pure incidence statements equivalent to **L**

- **ML**: Given three parallel lines, there is a line that intersects all three of them.
- $(\forall abc)(\exists u)\ \times(ab) \vee \times(bc) \vee \times(ca) \vee (\times(au) \wedge \times(bu) \wedge \times(cu))$
- **L**$(\exists\forall)$ There exist lines a and b, such that any line intersects a or b.

Pure incidence statements equivalent to **L**

- **ML**: Given three parallel lines, there is a line that intersects all three of them.
- $(\forall abc)(\exists u) \times(ab) \vee \times(bc) \vee \times(ca) \vee (\times(au) \wedge \times(bu) \wedge \times(cu))$
- **L**$(\exists\forall)$ There exist lines a and b, such that any line intersects a or b.
- $(\exists ab)(\forall g) \times (ga) \vee \times(gb)$

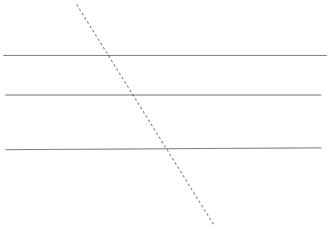

Figure: **ML**: There is a line meeting three given parallel lines

How were equivalences proved

- Synthetically: **ML** equivalent to **L** (over $\mathcal{A}$)

How were equivalences proved

- Synthetically: **ML** equivalent to **L** (over $\mathcal{A}$)
- By using Pejas: **L**($\exists\forall$) equivalent to **L** (over $\mathcal{A}$)

A universal statement in the language of line-intersections

- **ML**($\forall$) If the lines a_1, a_2, ad a_3 are pairwise parallel, then there is a permutation (i, j, k) of $(1, 2, 3)$ such that any line g which intersects a_i and a_j also intersects a_k.

A universal statement in the language of line-intersections

- **ML**($\forall$) If the lines a_1, a_2, ad a_3 are pairwise parallel, then there is a permutation (i, j, k) of $(1, 2, 3)$ such that any line g which intersects a_i and a_j also intersects a_k.
- **ML**$(\forall)_\lambda$: $\bigvee_{i=1}^{3} \times(a_i a_{i+1}) \vee (\times(ga_i) \wedge \times(ga_{i+1}) \to \times(ga_{i+2}))$

A universal statement in the language of line-intersections

- **ML**($\forall$) If the lines a_1, a_2, ad a_3 are pairwise parallel, then there is a permutation (i, j, k) of $(1, 2, 3)$ such that any line g which intersects a_i and a_j also intersects a_k.
- **ML**$(\forall)_\lambda$: $\bigvee_{i=1}^{3} \times(a_i a_{i+1}) \vee (\times(g a_i) \wedge \times(g a_{i+1}) \rightarrow \times(g a_{i+2}))$
- The statement **ML**$(\forall)_\lambda$ makes is weaker than that of **ML**($\forall$), for it states that, given three parallel lines a_1, a_2, and a_3, for any line g, there is a choice of i and j in $\{1, 2, 3\}$ such that, if g intersects a_i and a_j, then it intersects a_k as well, where $\{k\} = \{1, 2, 3\} \setminus \{i, j\}$, but it does not ensure that it will be the same i and j for any line g. However, the two versions are easily seen to be equivalent.

Ar as incidence stateent

- **S**: Given a line a and two distinct intersecting lines m and n, each different from a, there exists a line g which intersects a and m, but not n.

Ar as incidence stateent

- **S**: Given a line a and two distinct intersecting lines m and n, each different from a, there exists a line g which intersects a and m, but not n.
- We have seen that **L** is equivalent to a universal statement in the language of lines and line-intersections. Even if we enlarge that language to allow for point and line variables, and have the line-intersection predicate, as well as B and $\equiv$, there is no universal statement even in that language that would be equivalent to **Ar**.

Ar as incidence stateent

- **S**: Given a line a and two distinct intersecting lines m and n, each different from a, there exists a line g which intersects a and m, but not n.
- We have seen that **L** is equivalent to a universal statement in the language of lines and line-intersections. Even if we enlarge that language to allow for point and line variables, and have the line-intersection predicate, as well as B and $\equiv$, there is no universal statement even in that language that would be equivalent to **Ar**.
- There is no $\forall\exists$ statement σ, expressed in terms of the collinearity predicate L, with $\mathcal{A} \vdash \mathbf{S} \leftrightarrow \sigma$.

Axiom **S**

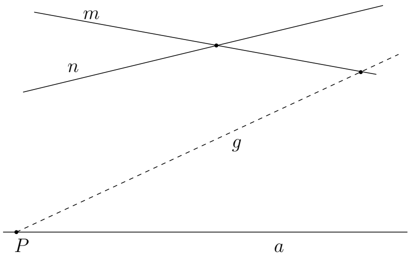

Figure: Axiom **S**

Equivalence between **Ar** and **S**

- The proof that $\mathcal{A} \vdash \mathbf{Ar} \Leftrightarrow \mathbf{S}$ has been carried out by relying heavily on Pejas's algebraic characterization. A synthetic proof is not known at present.

Splitting **P**, again

- **P** is thus equivalent to the conjunction fo **ML** and **S**, or of **L**(∃∀) and **S**. We have no synthetic proof for any of these equivalencies.

Splitting **P**, again

- **P** is thus equivalent to the conjunction fo **ML** and **S**, or of **L**(∃∀) and **S**. We have no synthetic proof for any of these equivalencies.
- When we express **S** in terms of points and the ternary collinearty relation L alone, then we get a ∀∃∀ statement.

Splitting **P**, again

- **P** is thus equivalent to the conjunction fo **ML** and **S**, or of **L**($\exists\forall$) and **S**. We have no synthetic proof for any of these equivalencies.
- When we express **S** in terms of points and the ternary collinearty relation L alone, then we get a $\forall\exists\forall$ statement.
- There is no $\forall\exists$ statement σ, expressed in terms of the collinearity predicate L, with $\mathcal{A} \vdash \mathbf{S} \leftrightarrow \sigma$.