Axiomatic problems in ordered geometry and the arithmetic of the even and odd

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The Pasch axiom

The arithmetic of the even and the odd

- The first axiom system for the arithmetic of the even and the odd
- The second axiom system for the arithmetic of the even and the odd

The linear order axioms

- The axiomatic framework is that of a very general two-dimensional theory of betweenness, the models of which will be referred to as *ordered planes*, axiomatized in terms of *points* as individual variables and the strict betweenness ternary predicate Z, with $Z(abc)$ to be read as 'b lies between a and c' the axiom system consisting of

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- $Z(cab) \wedge Z(abd) \to Z(cbd)$,
- $c \neq d \wedge Z(abc) \wedge Z(abd) \to (Z(bcd) \vee Z(bdc))$,

The plane order axioms

- The lower-dimension axiom, stating that there are three non-collinear points — here L stands for the collinearity predicate, defined by
  $L(xyz) :\Leftrightarrow Z(xyz) \vee Z(yzx) \vee Z(zxy) \vee x = y \vee y = z \vee z = x$;
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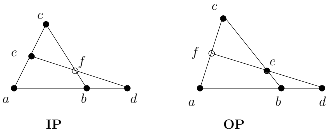
- and the Pasch axiom
  $$(\forall abcde)(\exists f)\, [\neg L(abc) \wedge Z(adc) \wedge \neg L(ace) \wedge \neg L(edb) \rightarrow (Z(afb) \vee Z(bfc)) \wedge L(edf)].$$

The inner and the outer form of the Pasch axiom

- To axiomatize higher-dimensional ordered spaces, Peano has introduced what we will refer to as the *outer* and the *inner* forms of the Pasch axiom



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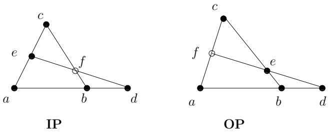

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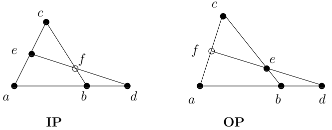

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- $(\forall abcde)(\exists f)\, \neg L(abc) \wedge Z(abd) \wedge Z(bec) \rightarrow Z(afc) \wedge Z(def)$,
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- **OP** is weaker than **IP**. That **OP** implies **IP** in the presence of the linear axioms with the unending one, was proved by O. Veblen (1904). That one cannot prove **OP** from **IP** was shown in a proof-theoretical manner in VP, 2010.

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- Sylvester-Gallai: Among n points, not all on one line, there are two, such that there is no other point from the set of those n points on that line.
- (Putnam problem, 1979): Given n red and n blue points, no three collinear, show that one can pair the red points with the blue points such that the resulting segments do not intersect.

The inner and the outer form of the Pasch axiom

- There is a whole book, W. A. Coppel, Foundations of convex geometry. Cambridge University Press, Cambridge, 1998, in which all the classical convex geometry results, Carathéodory, Helly, Rado, are rpoved with just **IP** and **OP**. What happens if we remove **OP**? Are all those results false in the presence of only **IP**?

Simplicity of Pasch's axiom

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- What can one say with 5 variables? Given three non-collinear points, *a*, *b*, *c*, a fourth point, not collinear with any of the first three, is in one of the following: (i) the interior of triangle *abc*; (ii) one of the three areas formed by the angles of the triangle pointing outside, or (iii) is in one of the areas moving out, determined by one side and the extensions pf the other two. Stating that a point is inside can be done in three ways, that it is in the areas (ii) also in three ways, that it is in the areas (iii) in only one way. There are thus 81 ways to express the fact that a fourth point must be in one of the triangle *abc* divides the plane into. Excluding symmetries, there are 15 such statements.

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The seven regions

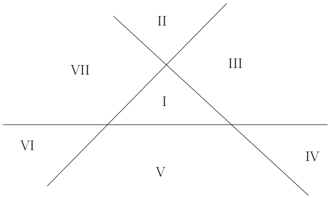


Figure: The seven regions

Ways of expressing location

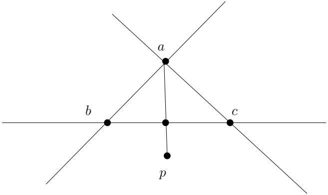
▶ 

Figure: Point opposite a vertex

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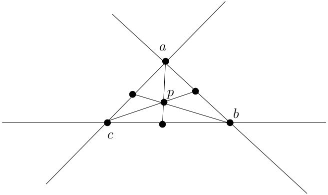

Figure: Point p inside triangle abc

Ways of expressing location

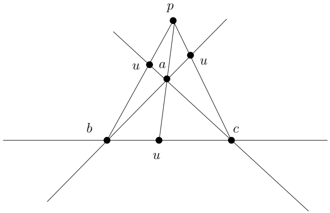

Figure: Point p in the angular area outside triangle abc

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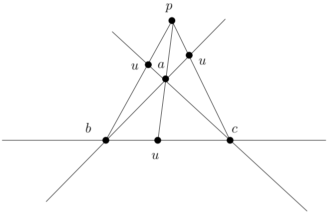

Figure: Point p in the angular area outside triangle abc

$$(Z(pau) \wedge Z(buc)) \vee (Z(pub) \wedge Z(cau)) \vee (Z(puc) \wedge Z(bau))$$

Are 5-variable sentences strong enough?

- Do they, taken together, imply the Pasch axiom? My belief is that this is not the case.

Splitting the Pasch axiom

- **WCBT** If p lies inside the angle $\widehat{bac}$ (which means that p lies on the same side of the line $\langle a, b \rangle$ as c, and on the same side of the line $\langle a, c \rangle$ as b (and two points u and v are said to be on the same side of line $\langle a, b \rangle$ if there is no point x belonging to line $\langle a, b \rangle$ with $Z(uxv)$)) then ap intersects the segment bc.

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- **PT** A line cannot intersect all three sides of a triangle.
- Pasch's axiom is equivalent to **WCBT** + **PT**. This is simpler than Pasch, for **WCBT** is $\forall\forall\forall\forall\exists$ and **PT** is universal. The Skolem function deriving from the Pasch axiom is a 5-variable function; the one associated with **WCBT** is a 4-variable function.

Theodorus' proof of irrationality

- Plato, through Theaetetus, tells us that: Theodorus was proving for us via diagrams something about powers, in particular about the three-foot-power and the five-foot-power — demonstrating that these are not commensurable in length with the one-foot-power; — and selecting each power individually in this way up to the seventeen-foot-power; but in this one for some reason he encountered difficulty.

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- The continued fraction of $\sqrt{19} = [4; \overline{2, 1, 3, 1, 2, 8}]$ has to long a period, and when done in the sand would lead to figures too small todraw. Others believe that Theodorus encountered a problem at $\sqrt{17}$.

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- Everywhere we encounter a mention of arithmetic in Plato's dialogues, we are told that arithmetic is the art of the even and the odd. *And since it's about odd and even, would it be anything except the art of arithmetic?*— Plato's dialogue *Protagoras*.

Why is the arithmetic of the even and the odd incapable of proving the irrationality of $\sqrt{17}$?

- The standard argument: Every odd square is a number of the form $8k+1$. Thus, to show, for odd d, that $m^2 = dn^2 \rightarrow m = 0$ has no solution in positive integers m and n, one notices that, if d is not of the form $8k+1$, then, since m^2 and n^2 are of that form, the equation cannot hold. If d is of the form $8k+1$, then that argument breaks down and there is no way to derive a contraction based on even and odd considerations. The first nonsquare odd d that is of the form $8k+1$ is 17.

What does it mean that one cannot prove the irrationality of $\sqrt{17}$ based on the arithmetic of the even and the odd?

- To provide a precise answer to that question we need a precise *arithmetic of the even and the odd*.

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- Two versions will be presented. The first was proposed by me, and I could show, by proof-theoretical methods, that $m^2 = 17n^2 \rightarrow m = 0$ cannot be proved from that axiom system. The second version, which is stronger, was put forward by Celia Schacht (2018), and there is still no proof of the non-derivability of $m^2 = 17n^2 \rightarrow m = 0$ from that axiom system.

The axiom system

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- $x + 0 = x \wedge x \cdot 0 = 0$

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- $x + 0 = x \wedge x \cdot 0 = 0$
- $x \cdot 1 = x$

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- $(x < y \wedge y < z) \rightarrow x < z$

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- $(0 < z \wedge x < y) \rightarrow x \cdot z < y \cdot z$

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- $x < y \rightarrow x + (y - x) = y$

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- $x < y \rightarrow x + (y - x) = y$
- $0 < 1 \wedge (x > 0 \rightarrow (x > 1 \vee x = 1))$
- $x > 0 \vee x = 0$
- So far, we have listed the axioms for discretely ordered rings. Without additional axioms, we can have arbitrarily long sequences of consecutive numbers none of which is even or odd.

Pythagorean Arithmetic

- To state the main axiom of the arithmetic of the even and the odd, we need two more binary operations, κ, and μ. The language in which our *Pythagorean Arithmetic* is expressed consists of 0, 1, $+$, $-$, $\cdot$, $<$, $\left[\frac{\cdot}{2}\right]$, κ, μ), together with the following two axioms: $x = \left[\frac{2x}{2}\right]$ and

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- $m = \kappa(m,n) \cdot \mu(m,n) \wedge n = \kappa(m,n) \cdot \mu(n,m)$
  $\wedge(\mu(m,n) = 2\left[\frac{\mu(m,n)}{2}\right] + 1 \vee \mu(n,m) = 2\left[\frac{\mu(n,m)}{2}\right] + 1)$
  This axiom accomplishes, with our modest means, what the fact that, for any positive integer n, there are non-negative integers $p(n)$ and $q(n)$ such that $n = 2^{p(n)}(2q(n)+1)$, does for natural numbers. Our $\kappa(m,n)$ plays here the role of $2^{\min\{p(n),p(m)\}}$, whereas $\mu(m,n)$ and $\mu(n,m)$ stand for $2^{p(m)-\min\{p(n),p(m)\}}(2q(m)+1)$ and $2^{p(n)-\min\{p(n),p(m)\}}(2q(n)+1)$ respectively.

Pythagorean Arithmetic

- One can show that Pythagorean Arithmetic does not prove $m^2 = 17n^2 \to 0$, although we do not have a concrete independence model.

A stronger arithmetic of the even and the odd (Schacht, 2018)

- The language of this axiom system consists of $0, 1, +, \cdot, <, -, \left[\frac{\cdot}{2}\right]$, and two unary operation symbols, τ and ω, whose meaning can be read off from the equation $n = \tau(n) \cdot \omega(n)$, with $\tau(n)$ a power of 2, and $\omega(n)$ odd. As a defined notion, meant to shorten formulas, we will use the unary predicate π_2, with $\pi_2(n)$ defined by $\tau(n) = n$, standing for "n is a power of 2."
  Our axiom system consists, in addition to all previously listed axioms in the language $0, 1, +, \cdot, <, -, \left[\frac{\cdot}{2}\right]$, of the following axioms:
- $\pi_2(n) \wedge a \cdot b = n \wedge a > 1 \rightarrow a = 2\left[\frac{a}{2}\right]$

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- $n < m \wedge \pi_2(m) \wedge \pi_2(n) \rightarrow \tau(m - n) = n$

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- The first axiom states that if $\pi_2(n)$ holds, then all the divisors of n that are greater than 1 must be even. We may thus think of $\pi_2(n)$ as a predicate describing the fact that n is a power of 2.The second axiom states that every positive integer can be written as the product of a power of 2 and an odd number. Motivated by the intuition that $2^a - 2^b = 2^b(2^{a-b} - 1)$ for $b < a$,the third axiom states that if $n < m$ are two powers of two, then n is a divisor of $m - n$.

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- μ and κ can be defined in this axiom system and the axiom describing these predicates can be proved in this stronger arithmetic.
- This corresponds exactly to the arithmetic of the even and the odd of the Pythagoreans. We do not have a proof of the non-derivability of $m^2 = 17n^2 \rightarrow m = 0$ from it.