On geometric rigidity of the rotation group and thin domains

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Motivation

In material science (elasticity, phase-transitions, etc.) the energy functional that should be minimized has the form

$$E(y) = \int_{\Omega} W(\nabla y(x))dx,$$

where $\Omega \subset \mathbb{R}^n$ and $y \in W^{1,p}(\Omega, \mathbb{R}^n)$. The function W has an energy well on a compact set $K \subset \mathbb{M}^{n \times n}$.

Question: Given some additional conditions on y, how close can one choose ∇y to the energy well of W?

Rigidity of the rotation group

$$SO(n) = \{R \in \mathbb{M}^{n\times n} \ : \ R \text{ is a proper rotation}\},$$

Theorem

Let $\Omega \subset \mathbb{R}^n$ be open and connected and let $1 < p < \infty$. Assume $v \in W^{1,p}(\Omega, \mathbb{R}^n)$. If

$$\nabla v(x) \in SO(n) \quad \text{for a.e.} \quad x \in \Omega,$$

then

$$\nabla v(x) = R = const.$$

(Reshetnyak, 1967)

Rigidity of the rotation group

Theorem

Let $\Omega \subset \mathbb{R}^n$ be open and connected and let $1 < p < \infty$. Assume $v, v_i \in W^{1,p}(\Omega, \mathbb{R}^n)$ such that

$$v_i \rightharpoonup v \quad \textit{in} \quad w^{1,p}(\Omega)$$

and

$$\operatorname{dist}(\nabla v_i, SO(n)) \to 0 \quad \textit{in measure.}$$

Then $\nabla v_i \to R$ in $L^p(\Omega)$ for some $R \in SO(n)$.

(Reshetnyak, 1967)

Proof of the rigidity

We have

$$\nabla v = \mathrm{cof}(\nabla v), \quad \text{a.e. in} \quad \Omega.$$

For smooth fields u an algebraic computation shows

$$\mathrm{div}(\mathrm{cof}(\nabla u)) = 0,$$

thus we get by a density argument

$$\Delta v = \mathrm{div}(\nabla v)) = 0,$$

in the sense of distributions. Weak harmonicity applies strong harmonicity, thus v is smooth.

$$0 = \frac{1}{2}\Delta(|\nabla v|^2 - n) = \nabla v \cdot \Delta(\nabla v) + |D^2 v|^2 = |D^2 v|^2,$$

thus $D^2 v = 0$.

Two-well rigidity

Theorem

Let $\Omega \subset \mathbb{R}^n$ be open and connected. Assume $v \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ satisfies

$$\nabla v(x) = A, \quad a.e. \ in \quad \Omega_A,$$

and

$$\nabla v(x) = B, \quad a.e. \ in \quad \Omega_B,$$

where Ω_A and Ω_B are disjoint with $|\Omega_A|, |\Omega_B| > 0$, $\Omega_A \cup \Omega_B = \Omega$. Then $A - B = a \otimes b$ for some $a, b \in \mathbb{R}^n$.

(Ball, James 1987)

Example: $v(x) = Bx + ah(x \cdot b)$, where $h \colon \mathbb{R} \to \mathbb{R}$ is Lipshitz with $h'(t) \in \{0, 1\}$ a.e.

Three-well rigidity

Theorem

Let $\Omega \subset \mathbb{R}^n$ *be open and connected. Assume* $v \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ *satisfies*

$$\nabla v(x) = A, \quad a.e. \ in \quad \Omega_A,$$

$$\nabla v(x) = B, \quad a.e. \ in \quad \Omega_B,$$

$$\nabla v(x) = C, \quad a.e. \ in \quad \Omega_C,$$

where Ω_A, Ω_B, *and* Ω_C *are disjoint with* $|\Omega_A|, |\Omega_B|, |\Omega_C| > 0$, $\Omega_A \cup \Omega_B \cup \Omega_C = \Omega$. *If* $\mathrm{rank}(A - B, B - C, C - A) \geq 2$, *then* ∇v *is constant in* Ω.

(Sverak, 1991)

Four and five-well rigidity

Four-well rigidity holds (Zhang, 1997.)

Five-well rigidity does not hold; (Kirchheim, 2003.)

Quantitative Geometric Rigidity

Theorem

Assume $\Omega \subset \mathbb{R}^3$ is open bounded and connected, and assume $1 < p \leq \infty$. There exists a constant $C = C(\Omega)$, depending only on Ω, such that for any $v \in W^{1,p}(\Omega)$, there exists a rotation $R \in SO(3)$, such that

$$\|\nabla v - R\|^p_{L^p(\Omega)} \leq C \int_\Omega \mathrm{dist}^p(\nabla v(x), SO(3)) dx.$$

(Friesecke, James, Müller, 2002.)

This result gives rise to derivation of plate theories and bending theories of shells in nonlinear elasticity.

Quantitative Rigidity for plates

Theorem

Assume $\omega \subset \mathbb{R}^2$ *is connected and compact, and assume* $1 < p \leq \infty$. *Denote* $\Omega = \omega \times (-h/2, h/2)$, *where* $h > 0$ *is a small parameter. Then there exists a constant* C, *depending only on* ω, *such that for any* $v \in W^{1,p}(\Omega)$, *there exists a rotation* $R \in SO(3)$, *such that*

$$\|\nabla v - R\|^p_{L^p(\Omega)} \leq \frac{C}{h^2} \int_\Omega \mathrm{dist}^p(\nabla v(x), SO(3))dx.$$

Moreover, the constant is asymptotically sharp.

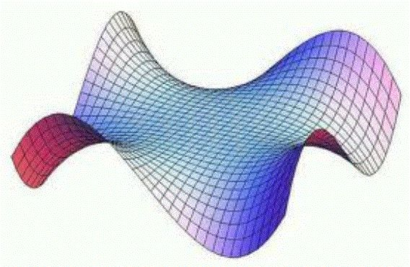
(Friesecke, James, Müller, 2002.)

Shells

Given a small number $h > 0$ and a smooth connected compact surface $S \subset \mathbb{R}^3$, a shell with mid-surface S and thickness h is the set

$$\Omega = \{x + t\vec{n}(x) \;:\; x \in S,\; t \in (-h/2, h/2)\},$$

where $\vec{n}(x)$ is the unit normal to S at the point x.



Rigidity for shells

Conjecture. For any connected compact C^1 surface S and any exponent $1 < p > \infty$, there exists $\alpha \in [1,2]$, such that there exists a constant $C(p,S)$, such that for any $v \in W^{1,p}(\Omega)$, there exists a rotation $R \in SO(3)$, such that

$$\|\nabla v - R\|^p_{L^p(\Omega)} \leq \frac{C}{h^\alpha} \int_\Omega \mathrm{dist}^p(\nabla v(x), SO(3)) dx.$$

Remark. The exponent α identifies the rigidity of Ω.

Thin domains

Let $g_1^h, g_2^h \colon S \to \mathbb{R}$ be Lipschitz, such that

$$c_1 h \leq g_i^h(x) \leq c_2 h \quad \text{and} \quad |\nabla g_i^h(x)| \leq c_3 h,$$

for $i = 1, 2$, $x \in S$ and for some constants $c_1, c_2, c_3 > 0$. Then the domain

$$\Omega = \{x + t\vec{n}(x) \ : \ x \in S, \ -g_1^h(x) < t < g_2^h(x)\}$$

is a shell with nonconstant thickness of order h, or just a thin domain.

Goal. Determine the rigidity of a given thin domain (shell).

Remark. All constants C will depend only on c_1, c_2, c_3 and S.

Korn's Inequalities

1. [Korn's first inequality without boundary conditions (KFINBC).]There exists a constant C_{NBC}, depending only on Ω such that for any $u \in W^{1,2}(\Omega)$, there exists a skew-symmetric matrix $A \in \mathrm{skew}(\mathbb{R}^n)$ such that

$$\|\nabla u - A\|_2 \leq C_{NBC}\|e(u)\|_2.$$

(Korn, 1908)

For $u \in W^{1,p}(\Omega, \mathbb{R}^n)$,

$$e(u) = \frac{1}{2}(\nabla u + (\nabla u)^T).$$

KFINBC=lineartization of GRE at the identity.

Korn's Inequalities

2. [Korn's first inequality with BC] There exists a constant C_{BC} depending only on Ω (and V) such that

$$\|\nabla u\|_2 \leq C_{BC}(\Omega, V)\|e(u)\|_2, \quad \text{for any} \quad u \in V \subset W^{1,2}(\Omega, \mathbb{R}^n)$$

.

3. [Korn's second inequality] There exists a constant C_2 depending only on Ω such that

$$\|\nabla u\|_2 \leq C_2(\Omega)(\|u\|_2 + \|e(u)\|_2), \quad \text{for any} \quad u \in W^{1,2}(\Omega, \mathbb{R}^n)$$

(Korn, 1908)

Conjecture on the rigidity of thin domains

Conjecture. Assume the mid-surface S of the thin domain Ω is C^1, connected and compact. Let K_G be the Gaussian curvature of S. Then

1. If $K_G > 0$, one has $C \sim h^{-1}$.
2. If $K_G < 0$, one has $C \sim h^{-4/3}$.

Results

Evidence. Assume the mid-surface S of the thin domain Ω is C^2, connected and compact (also some parametrization conditions). Let K be the Gaussian curvature of S. Then one has for zero (periodic) BC on the thin face of Ω, that

1. If $K > 0$, one has the scaling $C_{BC} \sim h^{-1}$,
   (H., 2018)
2. If $K < 0$, one has $C_{BC} \sim h^{-4/3}$,
   lower bound (H., 2018), Ansatz (Tovstik, 2001)
3. If $K \equiv 0$ with $\kappa_z \equiv 0$ and $\kappa_\theta > 0$, one has $C_{BC} \sim h^{3/2}$,
   (Grabovsky, H., 2018)

Remark: **The Ansätze for KFIBC work for GRE too:**

$$U = I + \epsilon u.$$

Shell parametrization

Consider a shell in the (r, θ, z) variables (θ and z are the principal directions):

$$\Omega = \left(-\frac{h}{2}, \frac{h}{2}\right) \times (0, \omega) \times (z_1(\theta), z_2(\theta)),$$

Let $R(\theta, z)$ be the position vector of the shell's mid-surface and

$$A_z = \left|\frac{\partial R}{\partial z}\right|, \qquad A_\theta = \left|\frac{\partial R}{\partial \theta}\right|.$$

Also

$$\kappa_z, \kappa_\theta - \text{principal curvatures}, \quad K = \kappa_z \kappa_\theta.$$

We assume that the surface S is regular and

$$\inf_{\theta \in [0,\omega]} [z_2(\theta) - z_1(\theta)] = l > 0, \qquad \sup_{\theta \in [0,\omega]} [z_2(\theta) - z_1(\theta)] = L < \infty,$$

$$\|z_1\|_{W^{1,\infty}[0,\omega]} + \|z_2\|_{W^{1,\infty}[0,\omega]} = Z < \infty.$$

The gradient

If $u = (u_r, u_\theta, u_z)$, then

$$\nabla u = \begin{bmatrix} u_{r,r} & \frac{u_{r,\theta} - A_\theta \kappa_\theta u_\theta}{A_\theta(1+r\kappa_\theta)} & \frac{u_{r,z} - A_z \kappa_z u_z}{A_z(1+r\kappa_z)} \\ u_{\theta,r} & \frac{A_z u_{\theta,\theta} + A_{\theta,z} u_z + A_\theta A_z \kappa_\theta u_r}{A_\theta A_z(1+r\kappa_\theta)} & \frac{A_\theta u_{\theta,z} - A_{z,\theta} u_z}{A_z A_\theta(1+r\kappa_z)} \\ u_{z,z} & \frac{A_z u_{z,\theta} - A_{\theta,z} u_\theta}{A_\theta A_z(1+r\kappa_\theta)} & \frac{A_\theta u_{z,z} + A_\theta A_z \kappa_z u_r + A_{z,\theta} u_\theta}{A_\theta A_z(1+r\kappa_z)} \end{bmatrix}.$$

Ansatz for $K < 0$

For $K < 0$ assume w, s and t ate smooth functions and choose

$$\begin{cases} u_r = w \\ u_\theta = v - r\left(\frac{w_{,\theta}}{A_\theta} - \kappa_\theta v\right) \\ u_z = s - r\left(\frac{w_{,z}}{A_z} - \kappa_z s\right). \end{cases}$$

Assume, next, the function $f(\theta, z)$ solves the transport equation

$$\frac{\kappa_\theta}{A_z^2}\left(\frac{\partial f}{\partial z}\right)^2 + \frac{\kappa_z}{A_\theta^2}\left(\frac{\partial f}{\partial \theta}\right)^2 = 0,$$

Denote $n(h) = \left[\frac{1}{n^{1/3}}\right]$, where $[x]$ is the integer part of x.

$$\begin{cases} w = n(h)\varphi(\theta, z)\sin(n(h)f(\theta, z)), \\ v = A_\theta \kappa_\theta \frac{\varphi(\theta,z)}{f_{,\theta}(\theta,z)}\cos(n(h)f(\theta, z)) \\ s = A_z \kappa_z \frac{\varphi(\theta,z)}{f_{,z}(\theta,z)}\cos(n(h)f(\theta, z)), \end{cases}$$

(Tovstik, 2001)

Ansatz for $K > 0$

For $K > 0$,

$$\begin{cases} u_r = W(\frac{\theta}{\sqrt{h}}, z) \\ u_\theta = -\frac{r \cdot W_{,\theta}\left(\frac{\theta}{\sqrt{h}}, z\right)}{A_\theta \sqrt{h}} \\ u_z = -\frac{r \cdot W_{,z}\left(\frac{\theta}{\sqrt{h}}, z\right)}{A_z}, \end{cases}$$

where W is a smooth function compactly supported on the mid-surface S.

The universal Korn interpolation inequality

Assume S can be cover by finitely many surfaces with the indicated property.

Theorem (H., 2020)

Assume the above conditions hold. There exists a constants $C > 0$, such that for any $u \in H^1(\Omega)$ one has

1. *[Korn's interpolation inequality]*

$$\|\nabla u\|_p^2 \leq C\left(\frac{\|u \cdot n\|_p \cdot \|e(u)\|_p}{h} + \|u\|_p^2 + \|e(u)\|_p^2\right).$$

2. *[Korn's second inequality]*

$$\|\nabla u\|_p^2 \leq \frac{C}{h}\left(\|u\|_p^2 + \|e(u)\|_p^2\right).$$

Both estimates are sharp (the Ansatz is exactly the same as for positive curvatures).

Universality of the KII

1. KII reduces KFINBC to a Korn-Poincaré inequalities, namely

$$\|u - Ax - b\| \leq ch^{-\beta}\|e(u)\|.$$

Then the KII implies a KFINBC with $h^{-\max(1+\beta,2\beta)}$.

2. The following estimates hold:

(i) If$K > 0$, then $\beta = 0$, $\max(1 + \beta, 2\beta) = 1$.

(ii) If $K < 0$, then $\beta = 1/3$, $\max(1 + \beta, 2\beta) = 4/3$.

(iii) If $K > 0$, then $\beta = 1/2$, $\max(1 + \beta, 2\beta) = 3/2$.

Key estimates

Lemma (1)

Let $h, l > 0$ *such that* $h < l/3$, *and let the Lipschitz functions* $\varphi_1, \varphi_2 \colon [0, l] \to (0, \infty)$ *and the constants* $C_1, C_2 > 0$ *be such that*

$$h \leq \varphi_i(y) \leq C_1 h, \quad |\nabla \varphi_i(y)| \leq C_2 h \quad \text{for all} \quad y \in [0, l], \quad i = 1, 2.$$

Denote the thin domain

$$D = \{(x, y) \in \mathbb{R}^2 \ : \ y \in (0, l), x \in (-\varphi_1(y), \varphi_2(y))\}$$

that has a thickness of order h. *Then there exists a constant* $c > 0$, *depending only on* C_1 *and* C_2, *such that any harmonic function* $w \in C^2(D)$ *fulfills the inequality*

$$\|w_y - a\|_{L^2(D)} \leq \frac{cl}{h} \cdot \|w_x\|_{L^2(D)},$$

where $a = \frac{1}{|D|} \int_D w_y$ *is the average of* w_y *over* D.

Key estimates

Lemma (2)

Let $h, l > 0$ *such that* $h < l/3$, *and let the Lipschitz functions* $\varphi_1, \varphi_2 \colon [0, l] \to (0, \infty)$ *and the constants* $C_1, C_2 > 0$ *be such that*

$$h \leq \varphi_i(y) \leq C_1 h, \quad |\nabla \varphi_i(y)| \leq C_2 h \quad \textit{for all} \quad y \in [0, l], \quad i = 1, 2.$$

Denote the thin domain

$$D = \{(x, y) \in \mathbb{R}^2 \; : \; y \in (0, l), x \in (-\varphi_1(y), \varphi_2(y))\}$$

that has a thickness of order h. *Then there exists a constant* $C > 0$, *depending only on* C_1 *and* C_2, *such that any harmonic function* $w \in C^2(D)$ *fulfills the inequality*

$$\|w_y\|_{L^2(D)} \leq C \left(\frac{1}{h} \cdot \|w\|_{L^2(D)} \|w_x\|_{L^2(D)} + \frac{1}{l^2} \|w_x\|_{L^2(D)} + \|w_x\|_{L^2(D)} \right).$$

The universal GR interpolation inequality

Theorem (H., 2020)

Assume $S \subset \mathbb{R}^3$ *is a compact bi-Lipschitz surface and assume* Ω *is a thin domain around* S *with thickness of order* h*. Then there exists a constants* $C > 0$*, such that for any vector field* $u \in W^{1,p}(\Omega)$*, any rotation* $R \in SO(3)$*, and any vector* $b \in \mathbb{R}^3$*, there holds:*

$$\|\nabla u - R\|_p^2 \leq C\Big(\frac{\|(u - Rx - b)\|_p \cdot \|\mathrm{dist}(\nabla u, SO(3))\|_p}{h} + \|u - Rx - b\|_p^2 + \|\mathrm{dist}(\nabla u, SO(3))\|_p^2\Big).$$

The estimate is sharp (the Ansatz is exactly the same as for positive curvatures).

Last slide

Thank you!