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Uniquely Decodable Codes via Augmented Sylvester-Hadamard Matrices

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Outline

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- Introduction
  - 5G and beyond
- Multiple access techniques
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5G and beyond

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- While 6G is in its early stage, the implementation of 5G around the globe is still ongoing.

Figure: Evolution from 1G to 5G

5G and beyond

The requirements of 5G are:

- High spectral and energy efficiency
- Massive connectivity
- Low latency

Some promising technologies are expect to address the requirement are:

- Massive multiple-input multiple-output (MIMO),
- Massive machine-type communication (mMTC),
- Millimeter-wave communications
- Non-orthogonal multiple access (NOMA)

Orthogonal multiple access (OMA)

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In the previous generations from 1G to 4G, the multiple access schemes were mostly characterized by orthogonal multiple access (OMA) techniques, where users are assigned orthogonal resources in either

- frequency, (frequency-division multiple access (FDMA))
- time, (time-division multiple access (TDMA))
- code, (code-division multiple access (CDMA))

However, the multiple access scheme in 5G is required to support a very wide range of use cases (e.g., massive numbers of low-power internet-of-things (IoT)).

Non-orthogonal multiple access (NOMA)

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These challenges can be addressed by the introduction of non-orthogonal multiple access (NOMA) techniques, which can be categorized into power-domain NOMA (PDM-NOMA) and code-domain NOMA (CDM-NOMA). A few of the strong contenders of CDM-NOMA are

- low-density spreading CDMA (LDS-CDMA)
- low-density spreading orthogonal frequency-division multiplexing (LDS-OFDM)
- sparse code multiple access (SCMA)
- pattern division multiple access (PDMA)
- multi-user shared access (MUSA)

Comparison with multiple access techniques

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- Direct sequence CDMA (DS-CDMA): symbols are spread in time. Multiple user spread over at same time and frequency.
- Multi-carrier CDMA (MC-CDMA): symbols are spread in frequency. Multiple user spread over same subcarriers at same time.
- LDS-OFDM: symbols are spread over large vectors most of whose elements are zero (sparse).

Comparison with multiple access techniques

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- SCMA: instead of repeating the same symbol on different subcarriers (as in LDS-OFDM), optimally coded symbols are spread on different subcarriers.

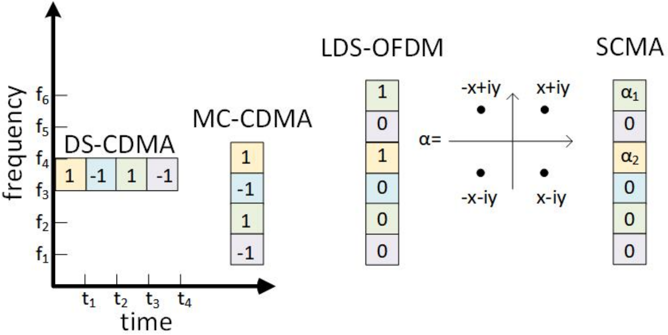

Figure:

Uniquely decodable (UD) codes

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- The antipodal code set $\mathbf{C} \in \{\pm 1\}^{L \times K}$ is uniquely decodable over signals $\mathbf{x} \in \{\pm 1\}^{K \times 1}$ or $\mathbf{x} \in \{0, 1\}^{K \times 1}$, $K > L$, if and only if, for any $\mathbf{x}_1 \neq \mathbf{x}_2$, $\mathbf{C}\mathbf{x}_1 \neq \mathbf{C}\mathbf{x}_2$ or, equivalently, $\mathbf{C}(\mathbf{x}_1 - \mathbf{x}_2) \neq \mathbf{0}_{L \times 1}$. We can rewrite the unique decodability necessary and sufficient condition as $\mathrm{Null}(\mathbf{C}) \cap \{0, \pm 2\}^{K \times 1} = \{0\}^{K \times 1}$ or in an equivalent manner as

$$\mathrm{Null}(\mathbf{C}) \cap \{0, \pm 1\}^{K \times 1} = \{0\}^{K \times 1}. \tag{1}$$

- Let $f_t(L)$ represent the maximum number of columns (signals) that matrix can have for a given L and still be uniquely decodable.

Uniquely decodable (UD) codes

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- Finding the overloaded UD class of codes for noiseless channel is directly related to coin-weighing problem, one of the problems that is discussed by Erdős and Rényi.
- What is the minimal number of weighings on an accurate scale to determine all false coins in a set of K coins.
- The choice of coins for a weighing must not depend on results of previous weighings.
- This problem was first introduced by Söderberg and Shapiro.

Uniquely decodable (UD) codes

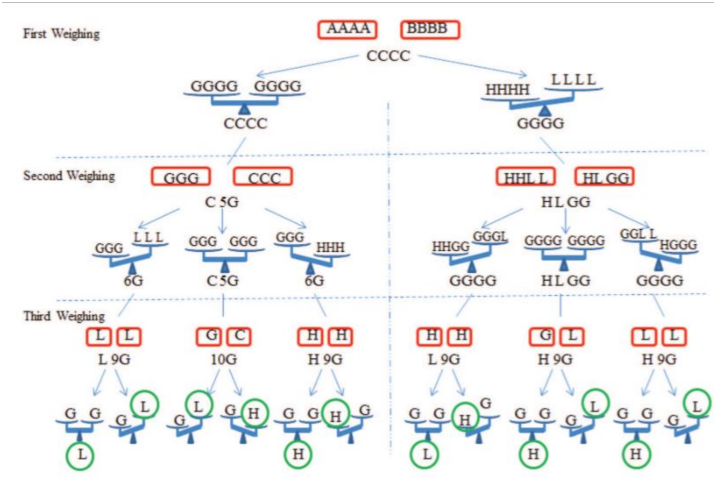

Figure:

Uniquely decodable (UD) codes

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- Lindström gives an explicit construction of $L \times \gamma(L+1)$ binary (alphabet $\{0,1\}$) and $L \times \gamma(L)+1$ antipodal (alphabet $\{\pm 1\}$) detecting matrices, where $\gamma(L)$ is the number of ones in the binary expansion of all positive integers less than L. An example, $\gamma(8) = 12$.
- He also proved that the lower bound in the case of $\mathcal{M} = \{0,1\}$ or $\{\pm 1\}$ is

$$\lim_{K \to \infty} \frac{\mathrm{f}_2(K) \log K}{K} = 2. \tag{2}$$

Uniquely decodable (UD) codes

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- Cantor and Mills constructed a class of $2^i \times (i+2)2^{(i-1)}$ ternary (alphabet $\{0, \pm 1\}$) detecting matrices for $i \in \mathbb{Z}^+$, which implies that in the case of $\mathcal{M} = \{0, \pm 1\}$ the lower bound is

$$\lim_{K \to \infty} \frac{\mathrm{f}_3(K) \log K}{K} \leq 2. \tag{3}$$

- The maximum number of vectors of the explicit constructions of binary, antipodal and ternary code sets are $K^b_{\max} = \gamma(L+1)$, $K^a_{\max} = \gamma(L) + 1$ and $K^t_{\max} = (i+2)2^{(i-1)}$, as shown in Table 1, Table 2 and Table 3, respectively.

Uniquely decodable (UD) codes

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Table: Binary Codes

Year	Authors	n	K	Decoder	
				Noiseless	AWGN
1963	Söderberg & Shapiro	L	$< \gamma(L+1)$	No	No
1964	Lindström	L	$\gamma(\mathbf{L}+\mathbf{1})^{\dagger}$	No	No
1966	Cantor & Mills	$2^k - 1$	$\mathbf{k2^{(k-1)}}$	No	No
1989	M. & Khachatrian	L	$\gamma(\mathbf{L}+\mathbf{1})$	Yes	No

† Code set constructions that achieve the maximum number of vectors $\mathbf{K}_{max}$ are presented in bold.

Uniquely decodable (UD) codes

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Table: Antipodal Codes

Year	**Authors**	**n**	**K**	**Decoder**	
				Noiseless	**AWGN**
1964	Lindström	L	$\gamma(\mathbf{L}) + \mathbf{1}$	No	No
1987	Khachatrian & M.	L	$\gamma(\mathbf{L}) + \mathbf{1}$	No	No
1995	Khachatrian & M.	2^k	$\mathbf{k2^{(k-1)}} + \mathbf{1}$	Yes	No
2012	Kulhandjian & Pados	2^k	$\mathbf{k2^{(k-1)}} + \mathbf{1}$	Yes	No
2018	Kulhandjian *et al.*	2^k	$\mathbf{k2^{(k-1)}} + \mathbf{1}$	Yes	Yes

Uniquely decodable (UD) codes

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Table: Ternary Codes

Year	Authors	n	K	Decoder	
				Noiseless	AWGN
1966	Cantor & Mills	2^k	$(\mathbf{k}+\mathbf{2})\mathbf{2}^{(\mathbf{k}-\mathbf{1})}$	No	No
1979	Chang & Weldon	2^k	$(\mathbf{k}+\mathbf{2})\mathbf{2}^{(\mathbf{k}-\mathbf{1})}$	Yes	No
1982	Ferguson	2^k	$(\mathbf{k}+\mathbf{2})\mathbf{2}^{(\mathbf{k}-\mathbf{1})}$	Yes	No
1984	Chang	2^k	$(\mathbf{k}+\mathbf{2})\mathbf{2}^{(\mathbf{k}-\mathbf{1})}$	No	No
1998	Khachatrian & M.	2^k	$(\mathbf{k}+\mathbf{2})\mathbf{2}^{(\mathbf{k}-\mathbf{1})}$	Yes	No
2012	M. & Marvasti	2^k	$2^{(k+1)}-1$	Yes	Yes
2016	Singh *et al.*	2^k	$2^{(k+1)}-2$	Yes	Yes
2018	Kulhandjian *et al.*	2^k	$2^{(k+1)}+2^{(k-2)}-1$	Yes	Yes

Uniquely decodable (UD) codes

k	L	Binary	Antipodal	Ternary
2	4	5	5	8
	5	7	6	
	6	9	8	
	7	12	10	
3	8	13	13	20
	9	15	14	
	10	17	16	
	11	20	18	
	12	22	21	
	13	25	23	
	14	28	26	
	15	32	29	
4	16	33	33	48
	17	35	34	
	18	37	36	

Formulation and Foundations

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- Code set $\mathbf{C} \in \{\pm 1\}^{L \times K}$ is uniquely decodable over signals $\mathbf{x} \in \{\pm 1\}^K$ if and only if for any $\mathbf{x}_1 \neq \mathbf{x}_2$, $\mathbf{C}\mathbf{x}_1 \neq \mathbf{C}\mathbf{x}_2$ or

$$\text{Null}(\mathbf{C}) \cap \{0, \pm 1\}^K = \{0\}^K. \tag{4}$$

- Consider $\mathbf{C} = [\mathbf{H}_L \mathbf{V}_L]$ where $\mathbf{H}_L$ Sylvester-Hadamard matrix of order $L = 2^p$, $p = 2, 3, ...,$ and $\mathbf{V}_L \in \{\pm 1\}^{L \times (K-L)}$. For any $\mathbf{z} \in \{0, \pm 1\}^K - \{0\}^K$ we must have

$$[\mathbf{H}_L \mathbf{V}_L]\, \mathbf{z} \neq \mathbf{0} \quad \text{or} \tag{5a}$$

$$\mathbf{H}_L \mathbf{z}_1 \neq -\mathbf{V}_L \mathbf{z}_2, \tag{5b}$$

$\mathbf{z} \triangleq \left[\mathbf{z}_1^T \mathbf{z}_2^T\right]^T$, $\mathbf{z}_1 \in \{0, \pm 1\}^L$, $\mathbf{z}_2 \in \{0, \pm 1\}^{K-L}$.

Formulation and Foundations

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- Questions:
  - (i) What is the maximum number of columns that we can append to a given Sylvester-Hadamard $\mathbf{H}_L$ and satisfy (5b)?
  - (ii) If we know the maximum number of columns $K - L$ that we can append, how do we design such a $\mathbf{V}_L \in \{\pm 1\}^{L \times (K-L)}$ to create the errorless code $\mathbf{C} = [\mathbf{H}_L \mathbf{V}_L]$?

Formulation and Foundations

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- **Proposition 1** Assume $\mathbf{z}_1 \neq \mathbf{0}^L$ in (5b). The code $\mathbf{C}$ is not uniquely decodable (not errorless) if and only if

$$\begin{aligned}
[\mathbf{v}_{0,0}, \ldots, \mathbf{v}_{N-1,0}]\, \mathbf{A}\mathbf{1}_N &= \mathbf{H}_4 \mathbf{A}_{H,0} \mathbf{1}_{N'} \\
[\mathbf{v}_{0,1}, \ldots, \mathbf{v}_{N-1,1}]\, \mathbf{A}\mathbf{1}_N &= \mathbf{H}_4 \mathbf{A}_{H,1} \mathbf{1}_{N'} \\
&\vdots \\
[\mathbf{v}_{0,M-1}, \ldots, \mathbf{v}_{N-1,M-1}]\, \mathbf{A}\mathbf{1}_N &= \mathbf{H}_4 \mathbf{A}_{H,M-1} \mathbf{1}_{N'}
\end{aligned} \tag{6}$$

for some $\mathbf{A} \in \{0, \pm 1\}^{N \times N}$, $\mathbf{A}_{H,i} \in \{0, \pm 1\}^{4 \times N'}$ $i = 0, \ldots, M-1$, that have at most one non zero entry in each column at the same position for all $i = 0, \ldots, M-1$ with values $[a_{0,j}, \ldots, a_{M-1,j}]^T \in \pm \mathbf{H}_M$ at column j, where $N, N' \in \mathbb{N}$, $\mathbf{1}_N = \{1\}^{N \times 1}$ and $\mathbf{1}_{N'} = \{1\}^{N' \times 1}$. □

Formulation and Foundations

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- **Proposition 2** Assume $\mathbf{z}_1 \neq \mathbf{0}^L$ in (5b). If

$$\begin{aligned} \beta_{0,0}\mathbf{v}_{0,0} \odot \ldots \odot \beta_{N-1,0}\mathbf{v}_{N-1,0} &= \alpha_0\mathbf{h}_j \\ \beta_{0,1}\mathbf{v}_{0,1} \odot \ldots \odot \beta_{N-1,1}\mathbf{v}_{N-1,1} &= \alpha_1\mathbf{h}_j \\ &\vdots \\ \beta_{0,M-1}\mathbf{v}_{0,M-1} \odot \ldots \odot \beta_{N-1,M-1}\mathbf{v}_{N-1,M-1} &= \alpha_{M-1}\mathbf{h}_j \end{aligned} \tag{7}$$

is not true for all $[\alpha_0, \ldots, \alpha_{M-1}]^T \in \mathbf{H}_M$, $\mathbf{h}_j \in \mathbf{H}_4$, $\beta_{i,j} \in \{0, 1\}$, $0 \leq i \leq N-1$, $0 \leq j \leq M-1$, $\beta_{0,0} = \beta_{0,1} = \ldots = \beta_{0,M-1}$, $\beta_{1,0} = \beta_{1,1} = \ldots = \beta_{1,M-1}$, ..., $\beta_{N-1,0} = \beta_{N-1,1} = \ldots = \beta_{N-1,M-1}$, then (5b) is satisfied. □

Formulation and Foundations

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- **Proposition 3** Assume $\mathbf{z}_1 = \mathbf{0}^L$ in (5b). The code $\mathbf{C}$ is not uniquely decodable if and only if

$$\begin{aligned} [\mathbf{v}_{0,0}, \ldots, \mathbf{v}_{N-1,0}]\,\mathbf{A}\mathbf{1}_N &= \mathbf{0} \\ [\mathbf{v}_{0,1}, \ldots, \mathbf{v}_{N-1,1}]\,\mathbf{A}\mathbf{1}_N &= \mathbf{0} \\ \vdots \quad & \quad \vdots \\ [\mathbf{v}_{0,M-1}, \ldots, \mathbf{v}_{N-1,M-1}]\,\mathbf{A}\mathbf{1}_N &= \mathbf{0} \end{aligned} \tag{8}$$

for some $\mathbf{A} \in \{0, \pm 1\}^{N \times N}$ where $N \in \mathbb{N}$, $\mathbf{1} = \{1\}^{4\times 1}$, $\mathbf{0} = \{0\}^{4\times 1}$ and $\mathbf{A}$ has at most one ± 1 entry in each column. □

Formulation and Foundations

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- **Proposition 4** Assume $\mathbf{z}_1 = \mathbf{0}^L$ in (5b). If

$$\begin{aligned} \beta_{0,0}\mathbf{v}_{0,0} \odot \ldots \odot \beta_{N-1,0}\mathbf{v}_{N-1,0} &= \alpha\mathbf{1} \\ \beta_{0,1}\mathbf{v}_{0,1} \odot \ldots \odot \beta_{N-1,1}\mathbf{v}_{N-1,1} &= \alpha\mathbf{1} \\ \vdots \quad & \quad \vdots \\ \beta_{0,M-1}\mathbf{v}_{0,M-1}\ldots \odot \beta_{N-1,M-1}\mathbf{v}_{N-1,M-1} &= \alpha\mathbf{1} \end{aligned} \tag{9}$$

is not true for all $\alpha \in \{\pm 1\}$, $\beta_{i,j} \in \{0,1\}$, $0 \leq i \leq N-1$, $0 \leq j \leq M-1$, $\beta_{0,0} = \beta_{0,1} = \ldots = \beta_{0,M-1}$, $\beta_{1,0} = \beta_{1,1} = \ldots = \beta_{1,M-1}$, $\beta_{N-1,0} = \beta_{N-1,1} = \ldots = \beta_{N-1,M-1}$, where $\odot$ operator denotes element by element multiplication of $\mathbf{v}_{i,j}$ vectors, then (5b) is satisfied. □

Uniquely decodable (UD) codes

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Table: Isomorphism $\varphi : G \mapsto \mathbb{F}_{2^4}$

Antipodal	Polynomial	Power
$\mathbf{h}_0$	0	0
$\mathbf{h}_2$	1	1
$\mathbf{h}_1$	α	α
$\mathbf{h}_0^-$	α^2	α^2
$\mathbf{a}_1$	α^3	α^3
$\mathbf{h}_3$	$\alpha + 1$	α^4
$\mathbf{h}_1^-$	$\alpha^2 + \alpha$	α^5
$\mathbf{a}_1^-$	$\alpha^3 + \alpha^2$	α^6
$\mathbf{a}_2$	$\alpha^3 + \alpha + 1$	α^7
$\mathbf{h}_2^-$	$\alpha^2 + 1$	α^8
$\mathbf{a}_3$	$\alpha^3 + \alpha$	α^9
$\mathbf{h}_3^-$	$\alpha^2 + \alpha + 1$	α^{10}
$\mathbf{a}_3^-$	$\alpha^3 + \alpha^2 + \alpha$	α^{11}
$\mathbf{a}_2^-$	$\alpha^3 + \alpha^2 + \alpha + 1$	α^{12}
$\mathbf{a}_0$	$\alpha^3 + \alpha^2 + 1$	α^{13}
$\mathbf{a}_0^-$	$\alpha^3 + 1$	α^{14}

Proof $K^a_{\max} = 13$ for $L = 8$

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- For the case when $L = 8$, we prove that the maximum number of columns we can append to $\mathbf{H}_8$ is actually $K^a_{\max} - L = 5$.
- we transform antipodal vectors into polynomials with integer coefficients $\mathbb{Z}[x]$, $\mathrm{F} : \{\pm 1\}^{m\times 1} \mapsto \mathbb{Z}[x]$. Those polynomials represent the row location and number of -1s or $+1$s in any antipodal $\mathbf{v} \in \{\pm 1\}^{m\times 1}$ with dimension m. Let the polynomial be

$$G(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \ldots + a_{m-1} x^{m-1}, \qquad (10)$$

where $a_i \in \mathbb{Z}$.

Proof $K^a_{\max} = 13$ for $L = 8$

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- Additions of vector $\mathbf{v}$ in vector space is equivalent to the addition of $\mathbb{Z}[x]$ in polynomial space. Each antipodal vectors, $\mathbf{v}_j \in \{\pm 1\}^{m \times 1}$, where $0 \leq j \leq 2^m - 1$, is mapped to the corresponding polynomials $G_n^j(x)$ and $G_p^j(x)$ to represent the $+1$ and -1 of the $\mathbf{v}_j$.
- As an example, for $m = 4$, the antipodal vector, $\mathbf{v}_{10} = [1, -1, 1, -1]^T$, is mapped to $G_n^{10}(x) = x^1 + x^3$ and $G_p^{10}(x) = x^0 + x^2$ polynomials.
- In order to visualize polynomial additions in 2-dimensional Euclidean space we can further transform $\mathbb{Z}[x]$ vector space into $\Lambda \subseteq \mathbb{Z}^2$ integer lattice, $\mathrm{H} : \mathbb{Z}[x] \mapsto \mathbb{Z}^2$.
- Let us define functions $\sigma_n^j(m) = G_n^j(m)$, $\sigma_p^j(m) = G_p^j(m)$, which are the evaluations of polynomials $G_n^j(x)$ and $G_p^j(x)$ at m, where m is the dimension of vector $\mathbf{v}_j$.

Proof $K^{a}_{\max} = 13$ for $L = 8$

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- By setting the x-axis and y-axis to be $G^{j}_{n}(x)$ and $G^{j}_{p}(x)$, we can build $\Lambda \subseteq \mathbb{Z}^2$ lattice points, since evaluations $\sigma^{j}_{n}(m)$ and $\sigma^{j}_{p}(m)$ for each antipodal $\mathbf{v}_j$ vectors are integers.
- Taking the above example of antipodal vectors having dimension of $m = 4$ the equivalent integer lattice points.

Proof $K^a_{\max} = 13$ for $L = 8$

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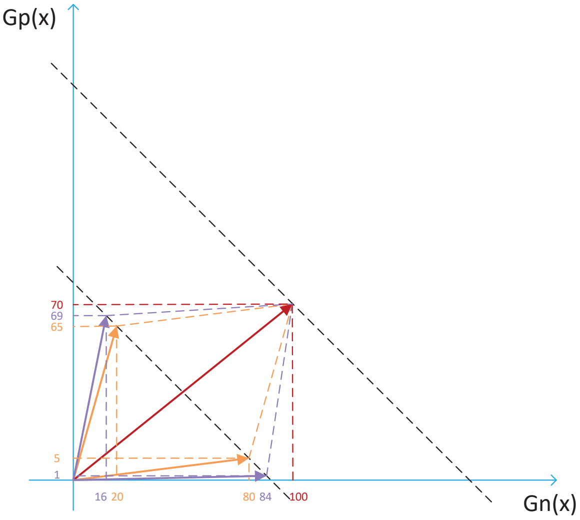

Figure: Addition of lattice points in $\Lambda \subseteq \mathbb{Z}^2$, $m = 4$.

The Maximum Size Errorless Code Set For $L = 8$ uOttawa

- For $L = 8$, describe the columns of $\mathbf{V}_L$, $\mathbf{v}_i$, $i = 0, ..., K - L - 1$, as two-dimensional finite-field vectors $\mathbf{f}_i = [f_{i,0} f_{i,1}]^T$, $f_{i,0}, f_{i,1} \in \{0, 1, \alpha, ..., \alpha^{14}\}$, $i = 0, ..., K - L - 1$.
- By Propositions 2 and 4, no vector subset of $[\mathbf{f}_0, \mathbf{f}_1, .., \mathbf{f}_{K-L-1}]$ can be of the form $[c, c]^T$, $[c^-, c]^T$, $[c, c^-]^T$, $[c^-, c^-]^T$ where $c \in \{0, 1, \alpha, \alpha^2, \alpha^4, \alpha^5, \alpha^8, \alpha^{10}\}$.

The Maximum Size Errorless Code Set For $L = 8$

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- By Propositions $1-4$, we establish that the maximum number of vectors that we can append to $\mathbf{H}_8$ and have uniquely decodable code $\mathbf{C}$ is five . Code Set example:

$$\mathbf{C}_{8\times 13} = \begin{bmatrix} + & + & + & + & + & + & + & + & - & + & + & - & + \\ + & - & + & - & + & - & + & - & + & + & - & + & - \\ + & + & - & - & + & + & - & - & + & - & + & + & + \\ + & - & - & + & + & - & - & + & + & - & - & + & + \\ + & + & + & + & - & - & - & - & + & + & + & - & - \\ + & - & + & - & - & + & - & + & + & + & + & + & + \\ + & + & - & - & - & - & + & + & + & + & + & + & - \\ + & - & - & + & - & + & + & - & + & + & + & + & - \end{bmatrix}$$

Figure: Errorless signature code set for an overloaded system with codelength $L = 8$ and code size $K = 13$.

Uniquely decodable (UD) codes

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- Our proposed design of antipodal matrices of $\mathbf{C}_{L\times K}$ are constructed as follows:

$$\mathbf{C}_{L\times K} = [\mathbf{H}_L \mathbf{V}_L], \tag{11}$$

where $\mathbf{V}_L \in \{\pm 1\}^{L\times(K-L)}$ and $\mathbf{H}_L$ is a Sylvester-Hadamard matrix.

- For the case when $L = 4$, we can augment the $\mathbf{H}_4$ matrix to form an UD ternary matrix, $\mathbf{C}_{4\times 5}$ as follows,

$$\mathbf{C}_{4\times 5} = \begin{bmatrix} + & + & + & + & - \\ + & - & + & - & + \\ + & + & - & - & + \\ + & - & - & + & + \end{bmatrix}. \tag{12}$$

Uniquely decodable (UD) codes

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- Let us consider UD code matrix for the case $L = 8$. We need to select a ternary UD matrix, $\mathbf{V}_8$, that when appended to $\mathbf{H}_8$ it will still satisfy the UD property. Moreover, $\mathbf{V}_8$ should also serve as the seed for generating $\mathbf{V}_L$ matrices for any L in recursive manner. In order to satisfy those requirement one possible option is to choose the following matrix,

$$\mathbf{V}_8 = \begin{bmatrix} \alpha^{13} & 1 & \alpha & \alpha^{13} & \alpha^{3} \\ 0 & 0 & 0 & \alpha^{13} & \alpha^{3-} \end{bmatrix}, \qquad (13)$$

Recursive Errorless Code Set Construction for $L > 8$

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- For $L > 8$ and $L = 2^p$, $p = 3, 4, \ldots$, generate $\mathbf{V}_L$ with dimensions $L \times [A(L) + 1]$

$$\mathbf{V}_L = \begin{bmatrix} \mathbf{V}_{L/2} & \mathbf{V}_{L/2} & \mathbf{R} \\ \mathbf{V}_{L/2} & \mathbf{V}^{-}_{L/2} & \mathbf{0}_{L/2} \end{bmatrix} \tag{14}$$

where $\mathbf{V}_{L/2}$ is the $L/2 \times [A(L/2) + 1 - L/2]$ matrix constructed in previous step and $\mathbf{R} = [\mathbf{r}_0, \ldots, \mathbf{r}_{M-1}]^T$ with $\mathbf{r}_i = \left[\bar{\mathbf{0}}^T_{4i}, \alpha^{13}, 1, \alpha, \mathbf{0}^T_{4(M-1-i)}\right]^T$, $0 \leq i \leq M - 1$, $M = L/8$, and $\bar{\mathbf{0}}_q$ the q-dimensional column vector with all elements zero from the extended field $GF(2^4)$ except at qth position which is 0^-.

Recursive Errorless Code Set Construction for $L > 8$ uOttawa

- For $L' = 0 (\mathrm{mod} 4)$ and $L' \neq 2^p$, $p = 1, 2, ...$

$$\mathbf{V}_{L'} = \begin{bmatrix} \mathbf{V}_L & \mathbf{0}_L \\ \mathbf{0}' & \mathbf{R}' \end{bmatrix} \tag{15}$$

where $L = 2^p$ is the largest integer such that $L < L' \leq 2L$, $L' = L + 4m$ ($m \in \{1, 2, 3, ...\}$), and $\mathbf{0}_L$ of dimensions $L \times 3m$ and $\mathbf{0}'$ of dimensions $m \times [A(L) + 1 - L]$ are all-zero matrices from the extended field $GF(2^4)$ and $\mathbf{R}' = \left[\mathbf{r}'_0, ..., \mathbf{r}'_{m-1}\right]^T$ with $\mathbf{r}'_i = \left[\mathbf{0}^T_{3i}, \alpha^{13}, 1, \alpha, \mathbf{0}^T_{3(m-1-i)}\right]^T$, $0 \leq i \leq m - 1$.

Minimum Distance of Code Sets

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- The Manhattan Distance between two L-dimensional vectors $\mathbf{y}_i$ and $\mathbf{y}_j$ for $i \neq j$ is equivalent to their (ℓ_1)-norm, which is defined as

$$d_L(\mathbf{y}_i, \mathbf{y}_j) = \sum_{t=1}^{L} |y_{i,t} - y_{j,t}|. \tag{16}$$

Then the general minimum Manhattan distance of received vectors for a given ternary code set can be formulated by

$$d_{min}(\mathbf{C}) = \underset{\substack{\mathbf{x}_i, \mathbf{x}_j \in \{\pm 1\}^{K \times 1} \\ \mathbf{y}_i = \mathbf{C}\mathbf{x}_i, \mathbf{y}_j = \mathbf{C}\mathbf{x}_j}}{\operatorname{argmin}} d_L(\mathbf{y}_i, \mathbf{y}_j). \tag{17}$$

Minimum Distance of Code Sets

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Theorem

Let $\mathfrak{C} \in \{\pm 1\}^{L\times K}$ *represent the set of all ternary matrices constructed by distinct* [a] *columns, then the minimum distance of the code set,* $\delta(\mathfrak{C})$, *is equal to* 2, *where*

$$\delta(\mathfrak{C}) = \underset{\mathbf{C}' \in \mathfrak{C}}{\text{argmin}}\, d_{min}(\mathbf{C}'). \tag{18}$$

[a] Not only the columns are required to be distinct but we assume any column multiplication with minus one should result in distinct columns as well.

Additive white Gaussian noise (AWGN)

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- In the overloaded (i.e., $K > L$) synchronous code-division multiple-access application of interest, each user multiplexes its own symbol by multiplying it with the signature and then transmitting it through the channel after carrier modulation.

$$\begin{aligned} \mathbf{y} &= \sum_{k=1}^{K} \mathbf{c}_k d_k x_k + \mathbf{n} \\ &= \mathbf{CDx} + \mathbf{n}, \end{aligned} \tag{19}$$

where d_k is the k-th user's amplitude, $x_k \in \mathcal{X}_k$ is the k-th user's symbol to be transmitted from the constellation alphabet, $\mathcal{X}_k$, $\mathbf{c}_k \in \{\pm 1\}^{K \times 1}$ is the spreading waveforms, $\mathbf{n} \in \mathbb{C}^{L \times 1}$ is an L-dimensional complex-valued AWGN vector with variance of σ^2 and $\mathbf{D}$ is a diagonal matrix with users' amplitude.

Low-complexity MUD

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- The objective of the receiver is the following; given the received vector $\mathbf{y}$ and $\mathbf{C}$ recover the user data $\hat{\mathbf{x}}$ such that the mean square error $E\{||\mathbf{x} - \hat{\mathbf{x}}||^2\}$ is minimized. It is known that obtaining the ML solution is generally NP-hard.
- The suboptimal detectors are more preferable in terms of complexity algorithm compared to ML. In this study, we consider two low-complexity detectors, MMSE-PIC detector, which is based on the MMSE and PIC criteria, and PDA.
- In case of flat Rayleigh fading and multipath fading channels we utilize the low-complexity multiuser detection (MUD) detectors.

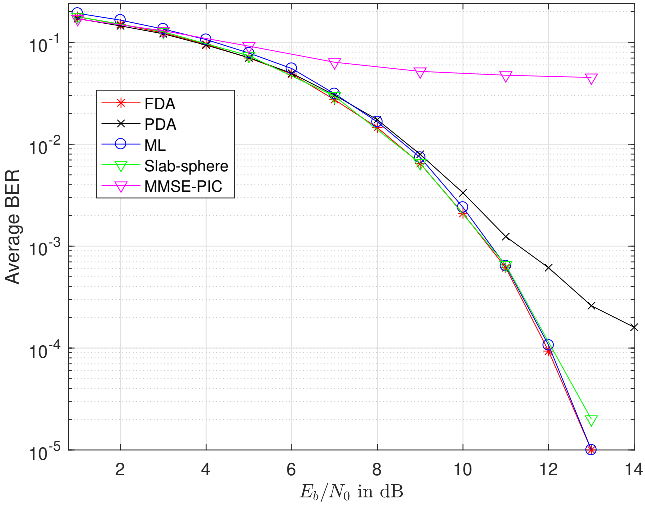
MMSE-PIC Detector

The computational complexity of the existing MMSE-PIC, MPA and PDA algorithms are compared with proposed algorithms in Table 5.

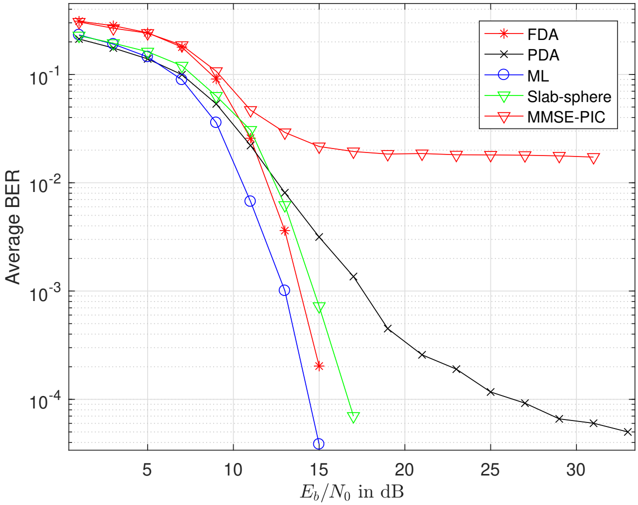
Table: Complexity Of Decoders

Algorithms	Complexity	Main procedures
NDA	$\mathcal{O}(L\log_2(L))$	Comparisons
FDA	$\mathcal{O}(LK\log_2(K))$	Comparisons
MMSE-PIC	$\mathcal{O}(LK^2)$	Inversion + multiplication
Slab-sphere	$\mathcal{O}(LK^2)$	Multiplication + comparisons
PDA	$\mathcal{O}(L^2K^2)$	Inversion + multiplication
ML	$\mathcal{O}(2^K)$	Multiplication + addition

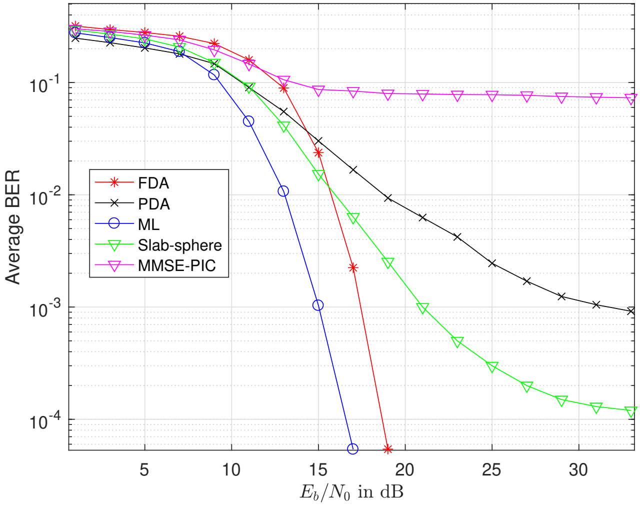
Results


FDA
PDA
ML
Slab-sphere
MMSE-PIC
Average BER
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
2
4
6
8
10
12
14
E_b/N_0 in dB

Results


FDA
PDA
ML
Slab-sphere
MMSE-PIC
Average BER
10^{-1}
10^{-2}
10^{-3}
10^{-4}
5
10
15
20
25
30
E_b/N_0 in dB

Results



Average BER
10^{-1}
10^{-2}
10^{-3}
10^{-4}
5
10
15
20
25
30
E_b/N_0 in dB
FDA
PDA
ML
Slab-sphere
MMSE-PIC

Conclusion

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- We proposed novel uniquely decodable (UD) antipodal code set recursive construction for overloaded (CDMA) systems.
- The proposed algorithm has a much lower computational complexity compared to the maximum likelihood (ML) decoder whose complexity may be prohibitive for even moderate code lengths.
- Trading flops for Hz : Uniquely decodable code sets can double (or more) the bandwidth at modest/moderate computational expense increase.

References

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Thank you!

Questions?